Linear Classifiers

Jaime Arguello
INLS 613: Text Data Mining
jarguell@email.unc.edu

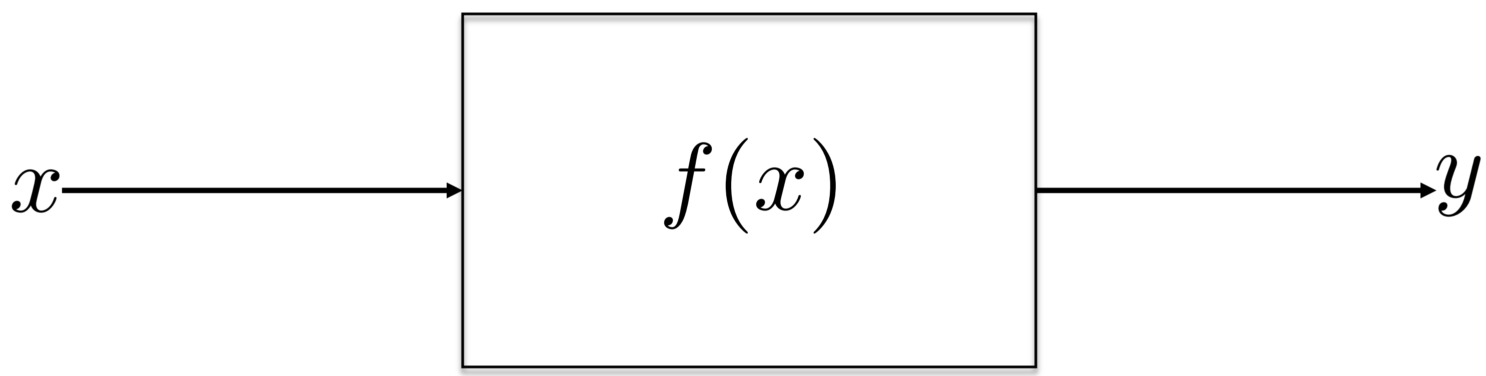
Overview

- Philosophical questions
- Derivatives: What are they good for?
- Linear regression
- Multiple linear regression
- Logistic regression

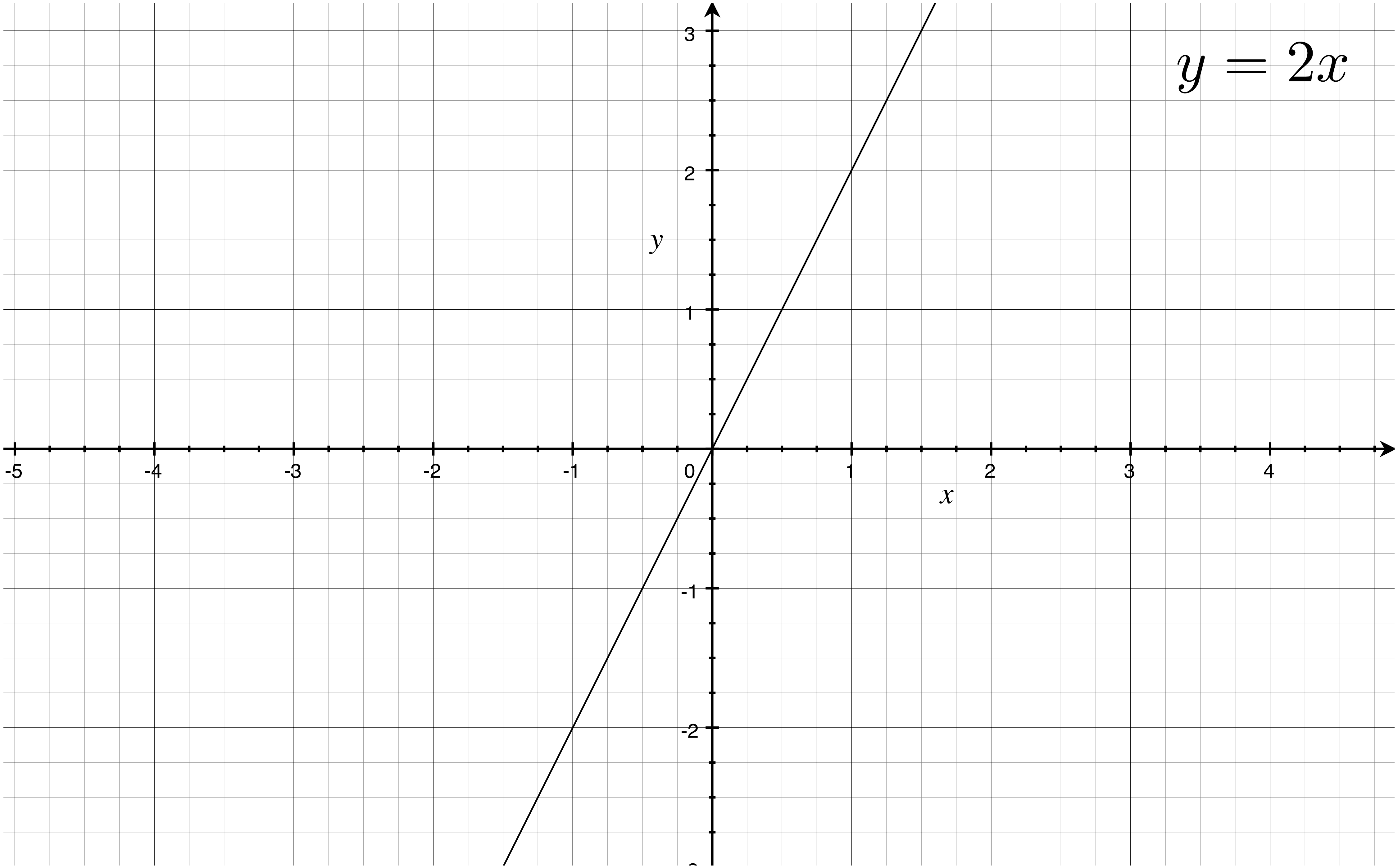
Philosophical Questions

- What would you do if …
- What does this have to do with linear classifiers?

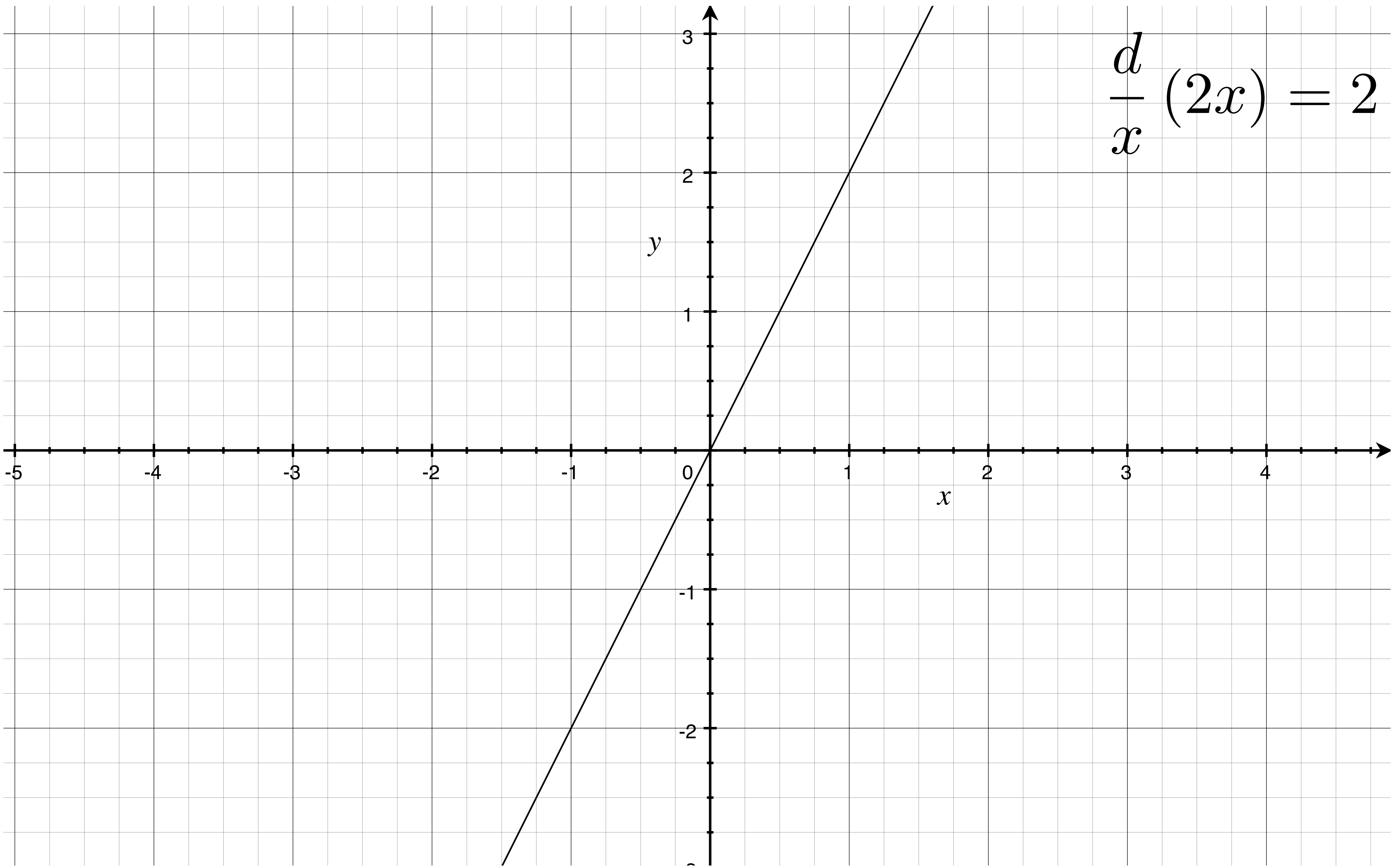
Functions



Derivatives

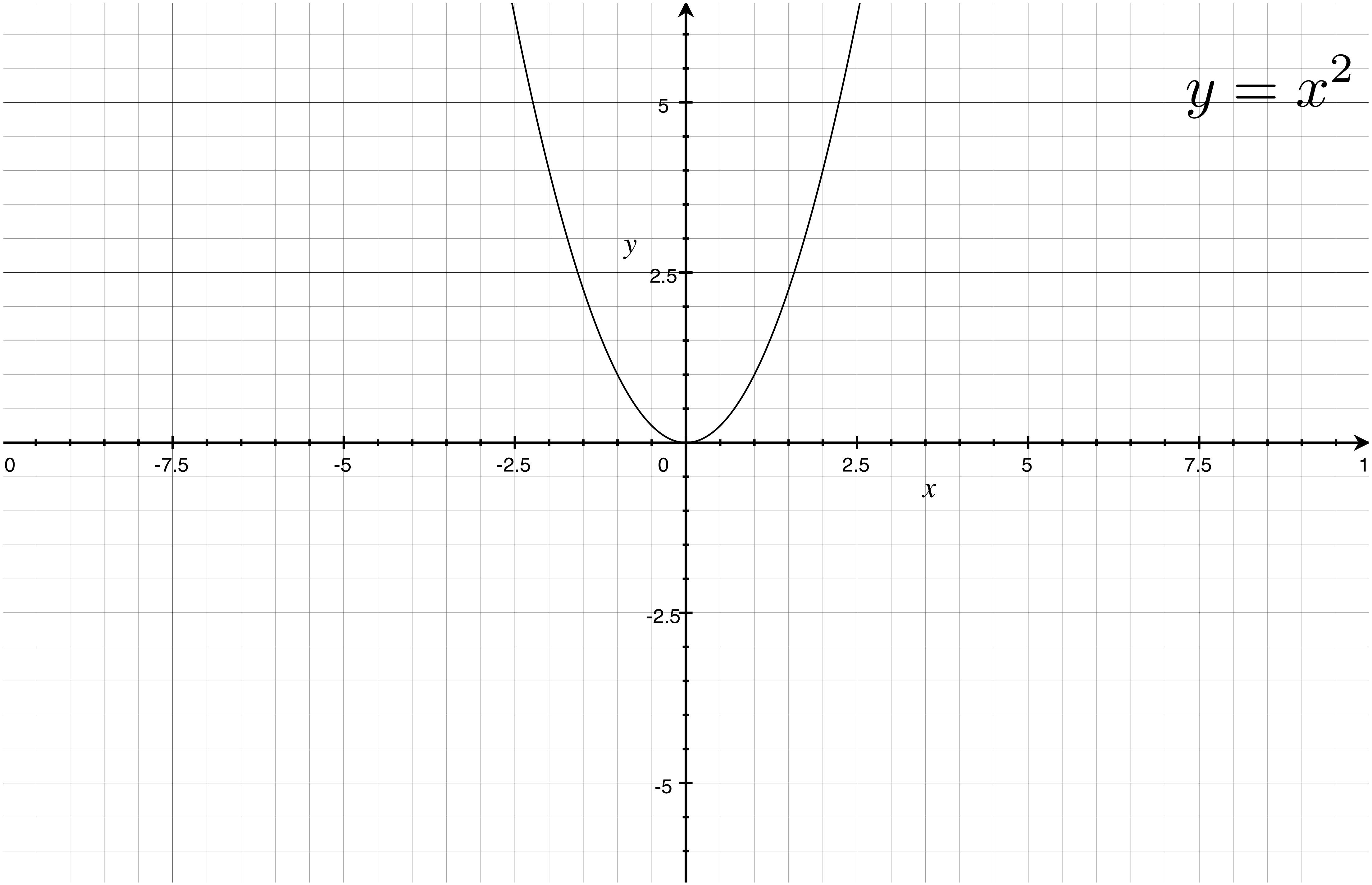


Derivatives

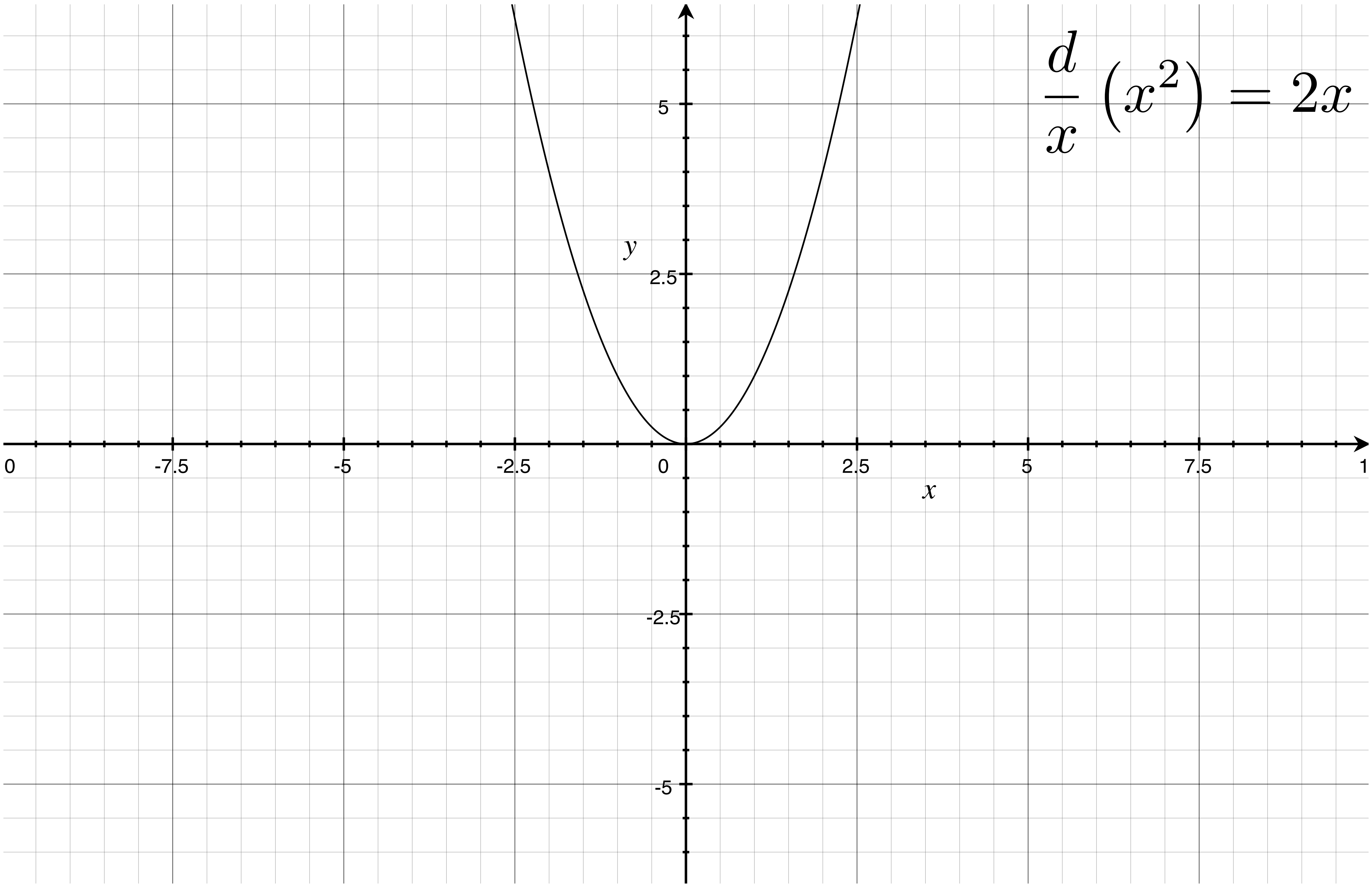


$$\frac{d}{x}(2x) = 2$$

Derivatives



Derivatives

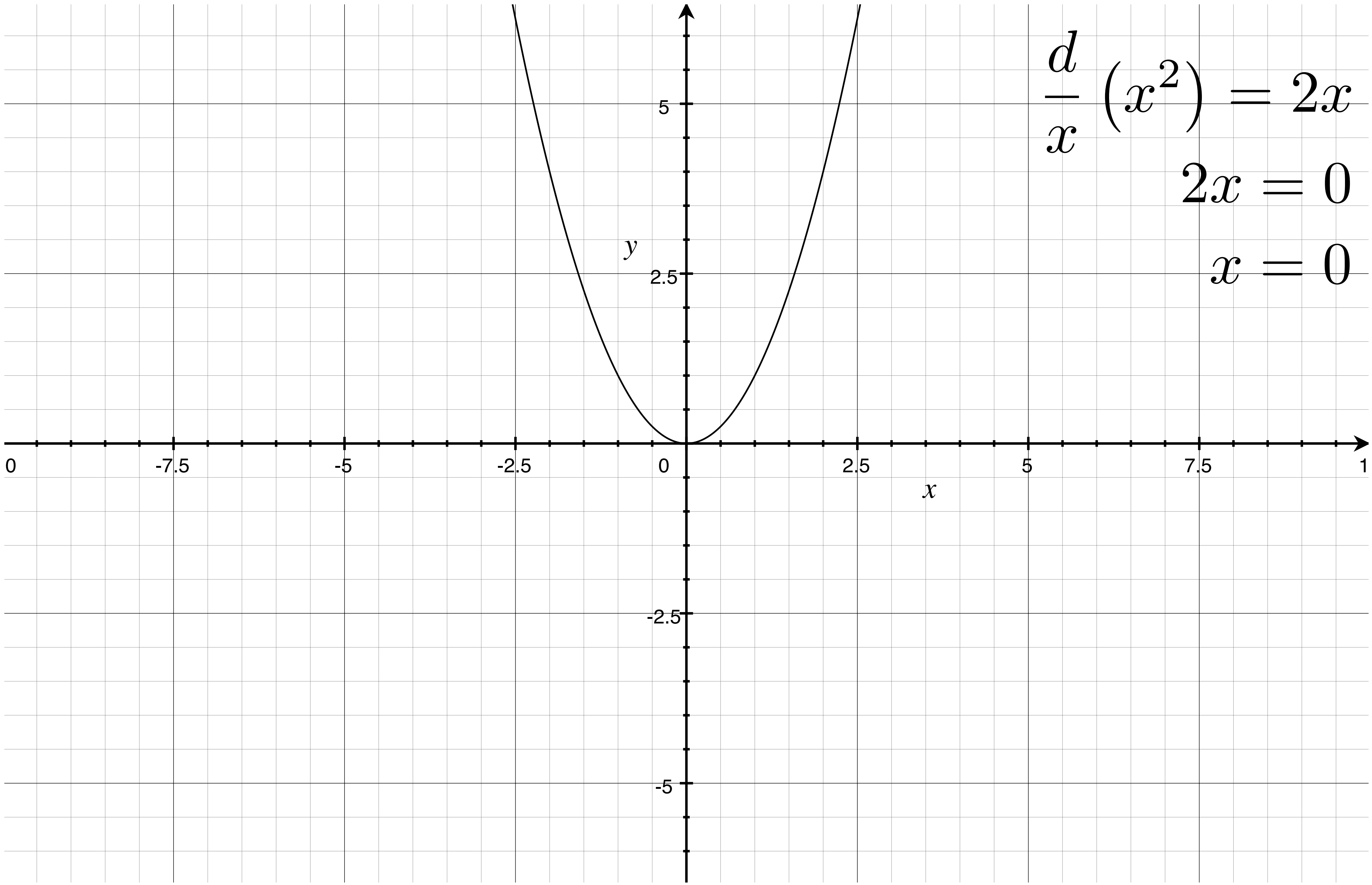


$$\frac{d}{x}\left(x^2\right) = 2x$$

Derivatives: What are they good for?

- The derivative of f(x) outputs the slope of f(x) for a particular value of x
- A point of which the slope is zero is a point at which f(x) is at its highest or lowest value.
- What does this have to do with machine learning?

Derivatives



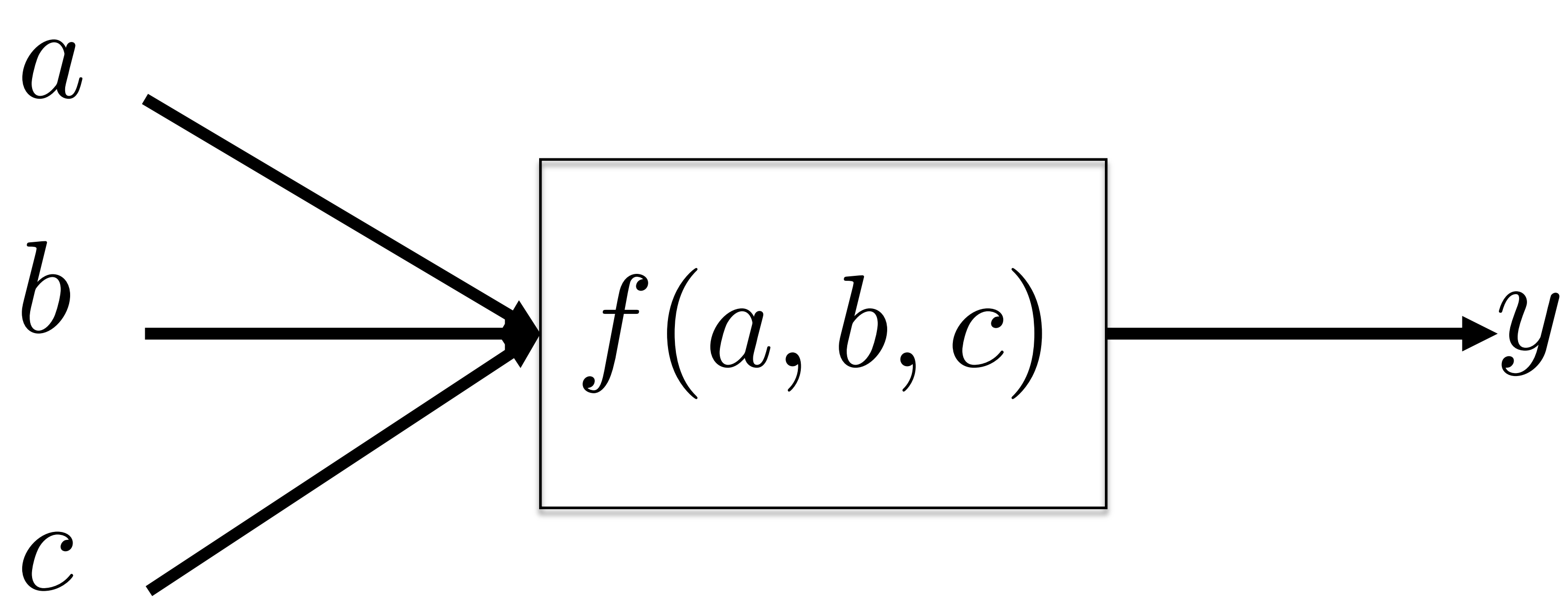
$$\frac{d}{x}\left(x^2\right) = 2x$$

$$2x = 0$$

$$x = 0$$

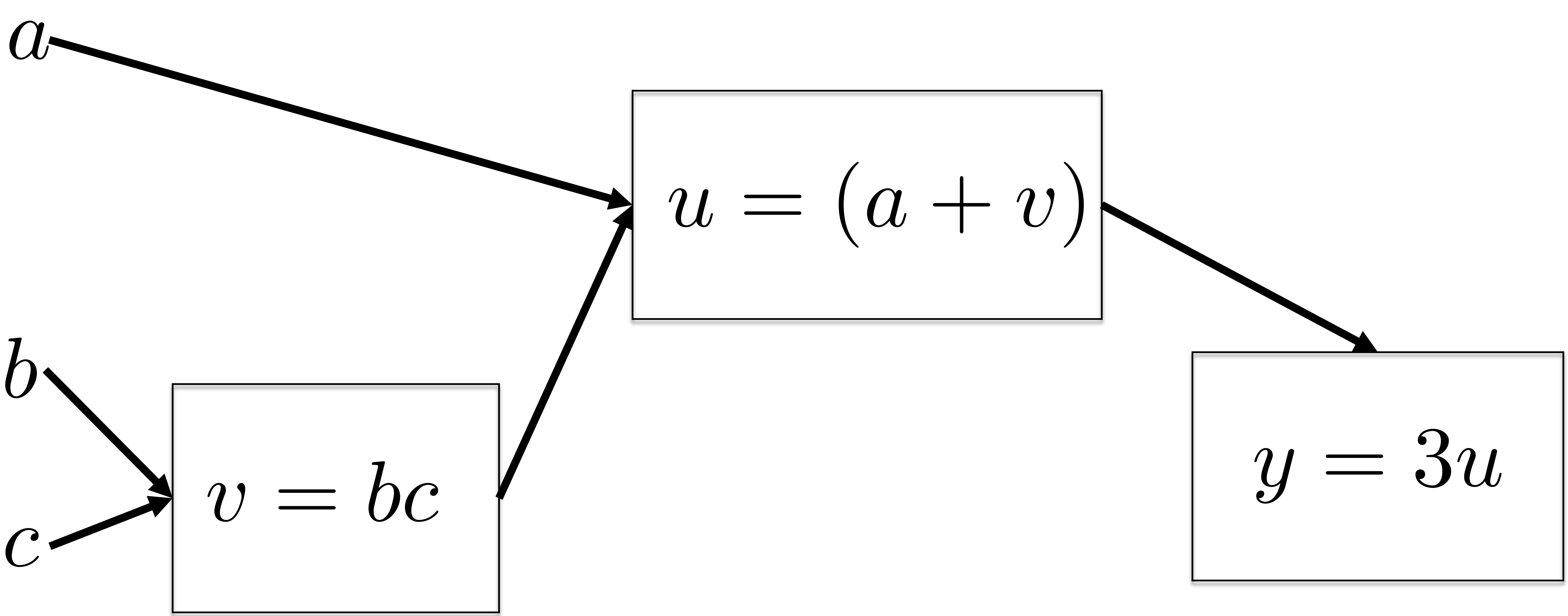
Computation Graphs

$$y = 3(a + bc)$$



Computation Graphs

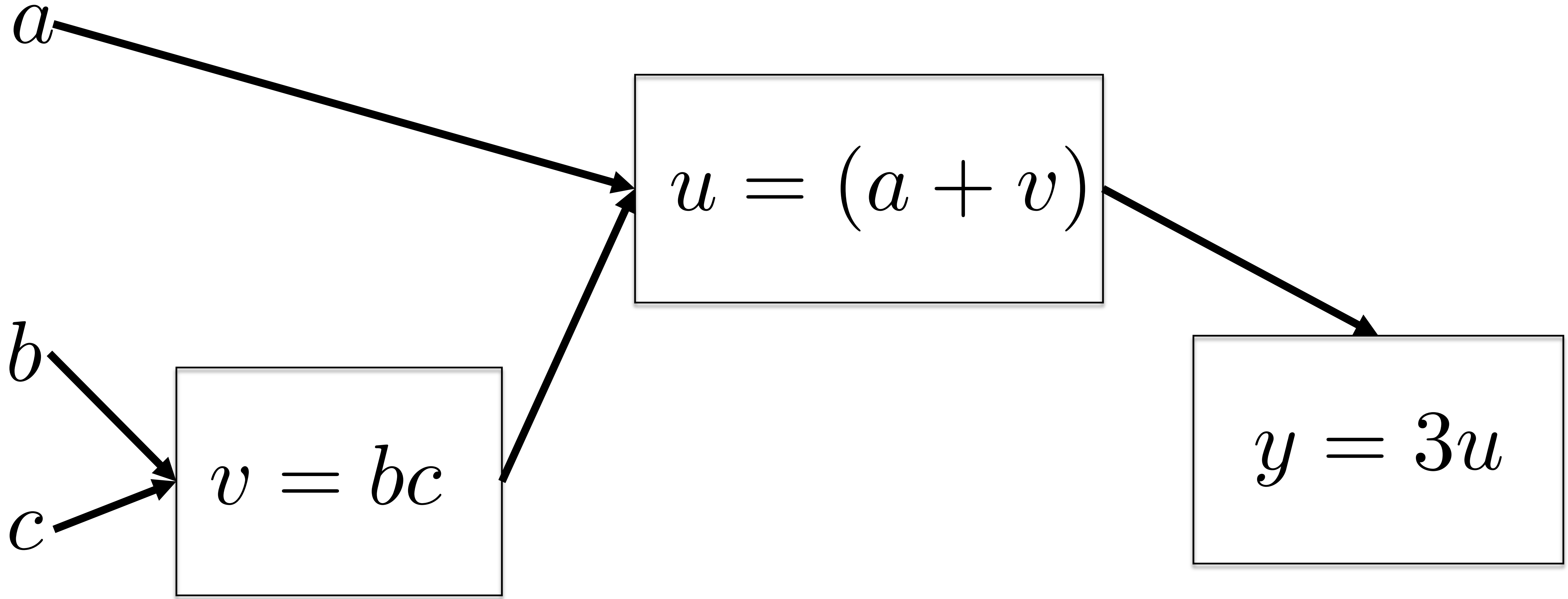
$$y = 3(a + bc)$$



Derivatives: Chain Rule

$$y = 3(a + bc)$$

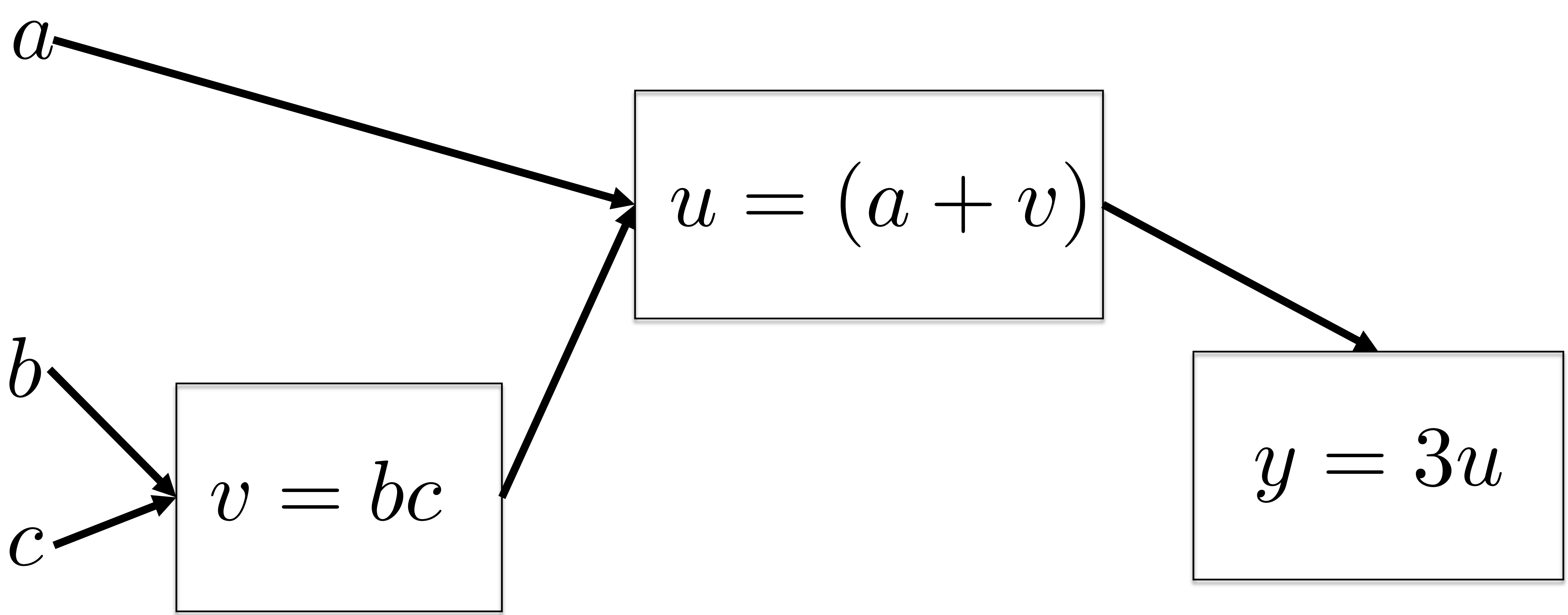
$$\frac{dy}{dc} = \frac{dv}{dc} \times \frac{du}{dv} \times \frac{dy}{du}$$



Derivatives: Chain Rule

$$y = 3(a + bc)$$

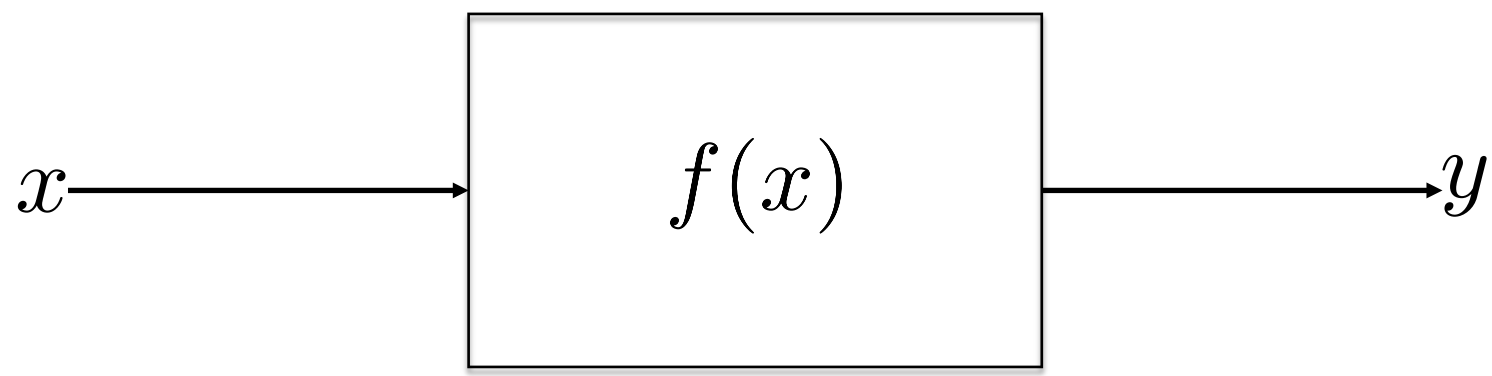
$$\frac{dy}{dc} = b \times 1 \times 3 = 3b$$



Overview

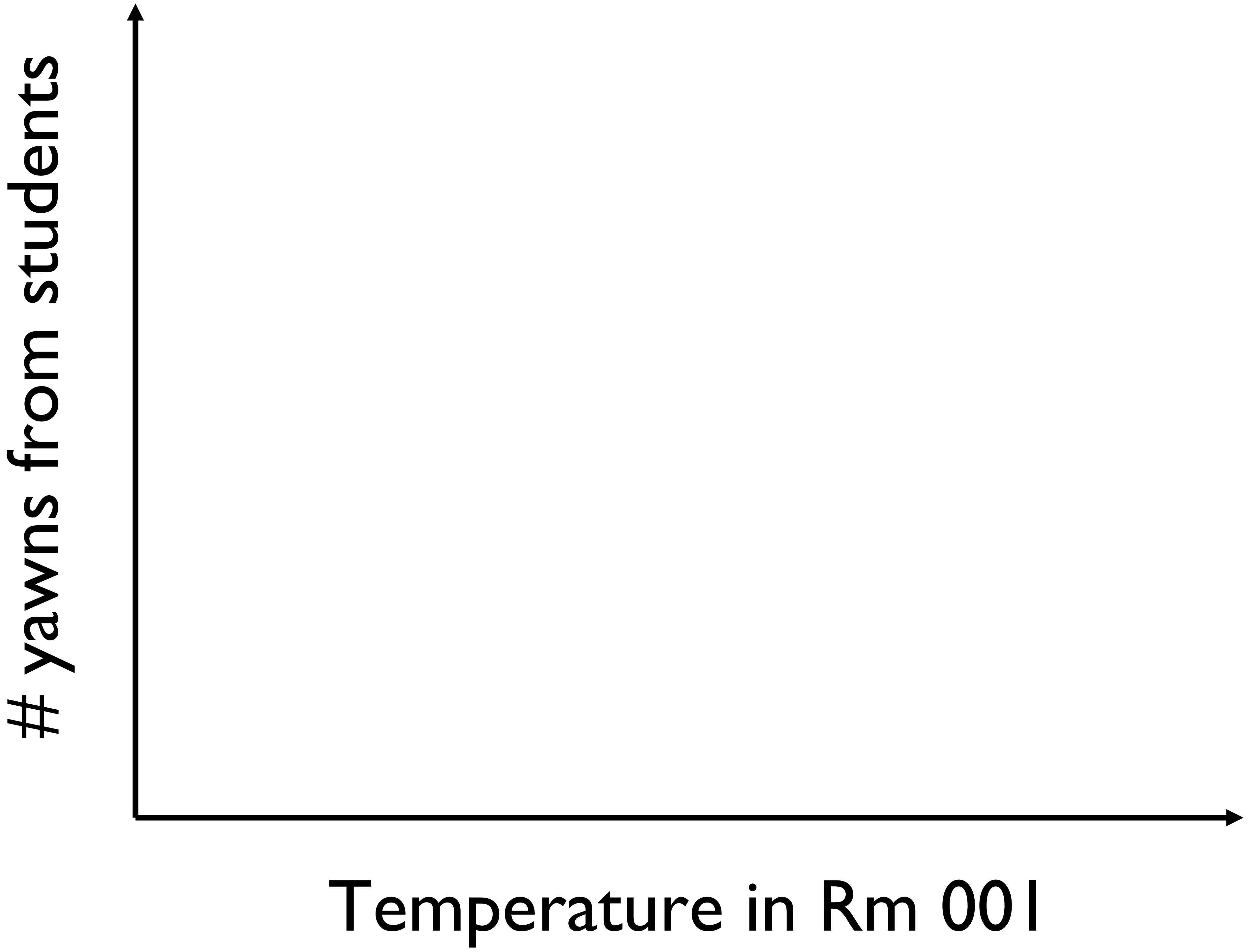
- Philosophical questions
- Derivatives: What are they good for?
- Linear regression
- Multiple linear regression
- Logistic regression

Linear Regression

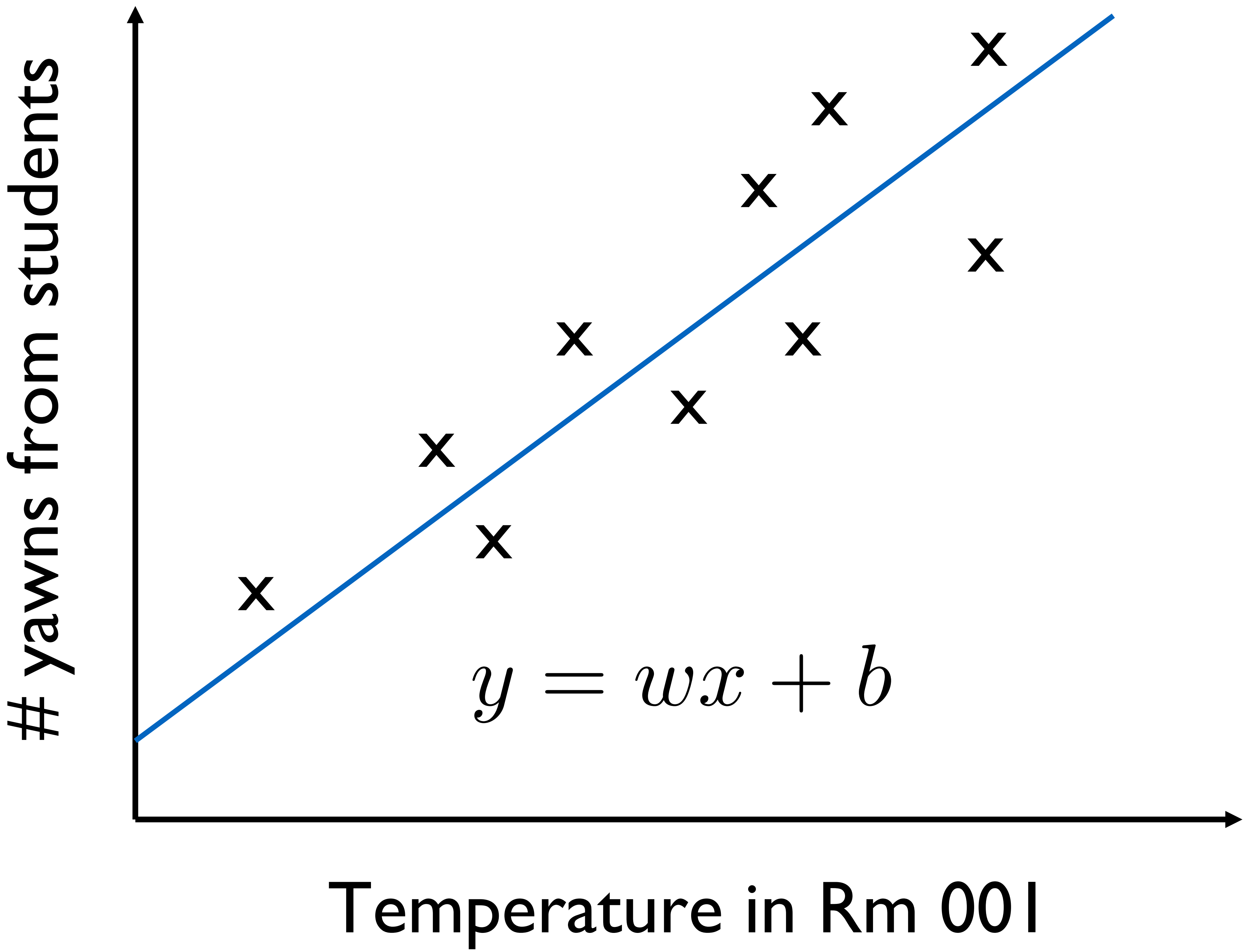


$$y = wx + b$$

Linear Regression



Linear Regression



Linear Regression: Training

$$y = wx + b$$

- **Input:** set of *m* training examples (x,y)
- Find the value of *w* and *b* that minimize the error:

$$\sum_{i=1}^{m} \left(y^{(i)} - \hat{y}^{(i)} \right)^2$$

Linear Regression: Training

$$y = wx + b$$

- Find the value of *w* and *b* that minimize the error:

$$\sum_{i=1}^{m} \left(y^{(i)} - \hat{y}^{(i)} \right)^2$$

$$\sum_{i=1}^{m} \left(y^{(i)} - wx^{(i)} - b \right)^2$$

Linear Regression: Training

$$y = wx + b$$

- Find the value of *w* and *b* that minimize the error:

$$\sum_{i=1}^{m} \left(y^{(i)} - wx^{(i)} - b \right)^2$$

- Take the derivative with respect to *w*, set it equal to 0, and solve for *w*.
- Take the derivative with respect to *b*, set it equal to 0, and solve for *b*.

Linear Regression: Training

- Find the value of *w* and *b* that minimize the error:

$$w = \frac{\frac{1}{m}\sum_{i=1}^{m}\left(x^{(i)} - \bar{x}\right)\left(y^{(i)} - \bar{y}\right)}{\sum_{i=1}^{m}\left(x^{(i)} - \bar{x}\right)^2}$$

$$b = \bar{y} - w\bar{x}$$

http://seismo.berkeley.edu/~kirchner/eps_120/Toolkits/Toolkit_10.pdf

Linear Regression: Training

- Find the value of *w* and *b* that minimize the error:

$$w = \frac{\frac{1}{m}\sum_{i=1}^{m}\left(x^{(i)} - \bar{x}\right)\left(y^{(i)} - \bar{y}\right)}{\sum_{i=1}^{m}\left(x^{(i)} - \bar{x}\right)^2}$$

Always positive!

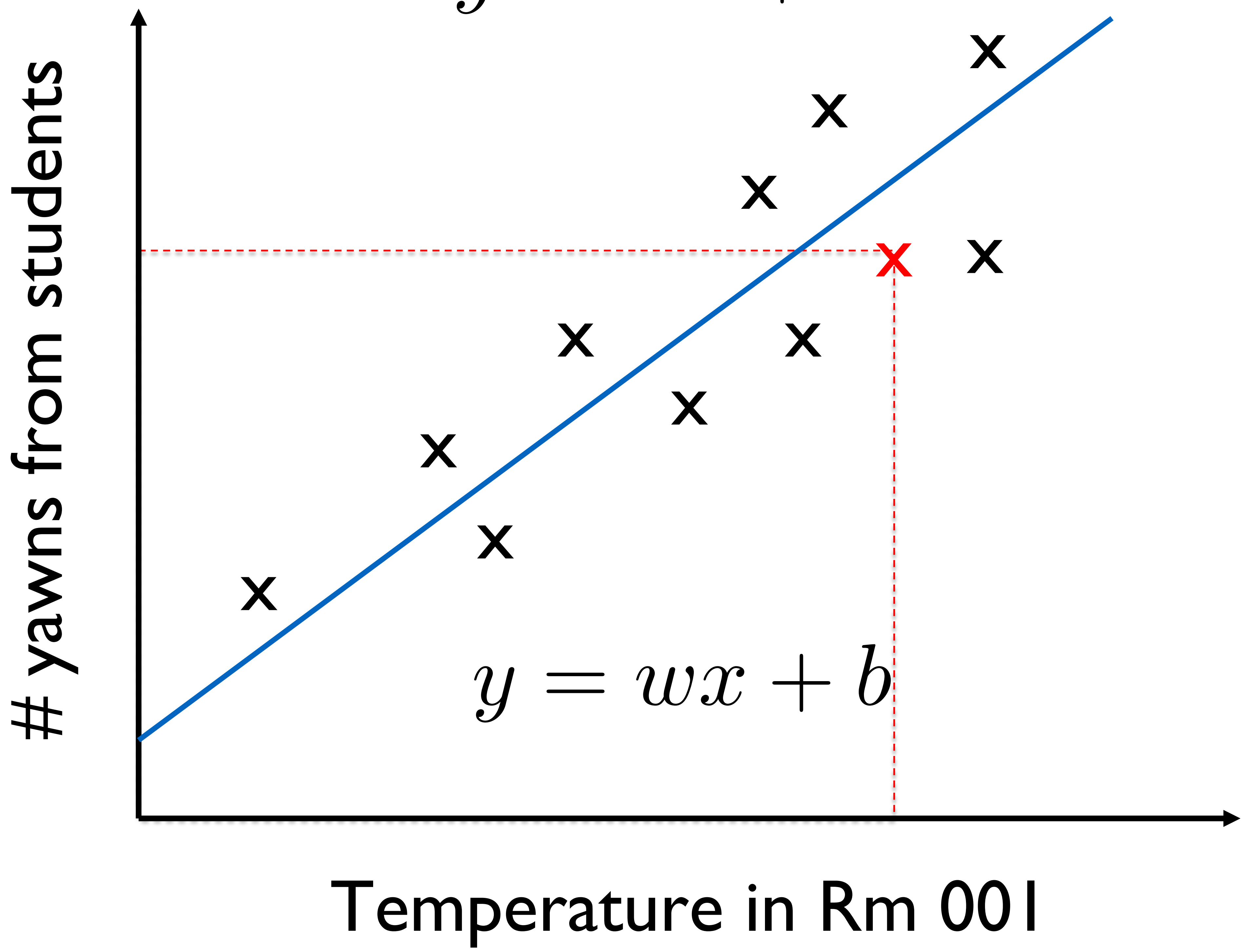
It depends!

$$b = \bar{y} - w\bar{x}$$

http://seismo.berkeley.edu/~kirchner/eps_120/Toolkits/Toolkit_10.pdf

Linear Regression: Prediction

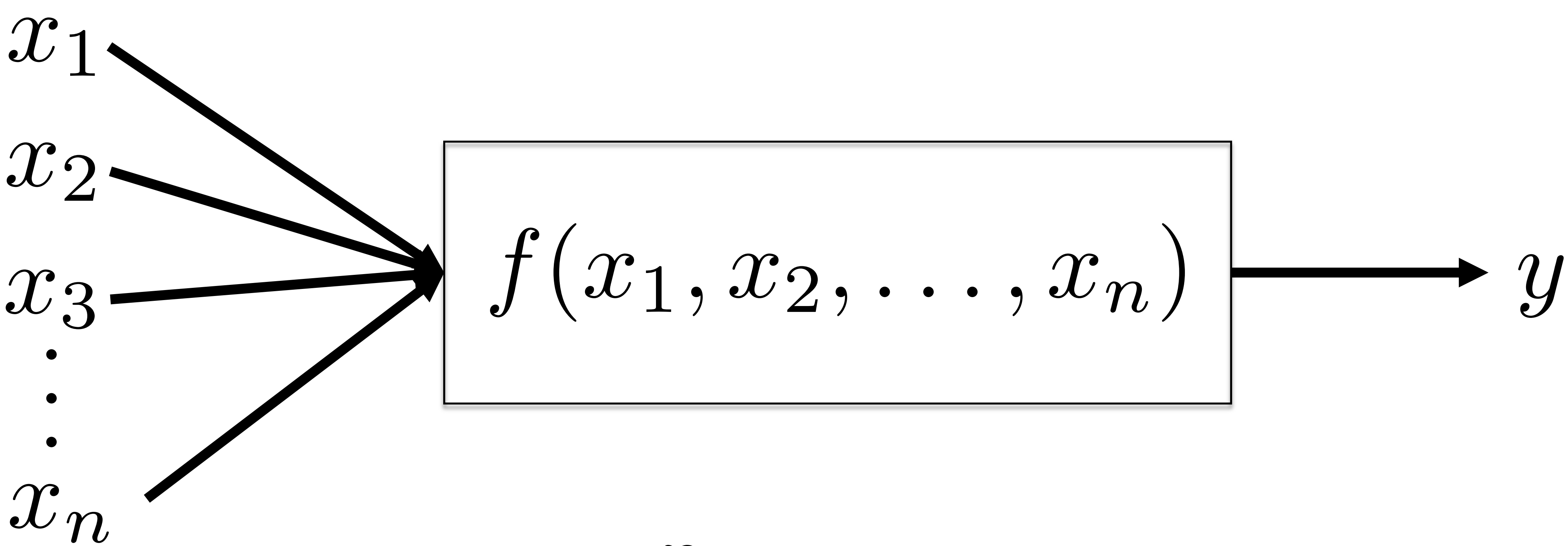
$$y = wx + b$$



Overview

- Philosophical questions
- Derivatives: What are they good for?
- Linear regression
- Multiple linear regression
- Logistic regression

Multiple Linear Regression



$$y = \sum_{j=1}^{n} (w_j x_j) + b$$

Multiple Linear Regression

Size (feet)	No. of bedrooms	No. of floors	Age (years)	Price (x$1000)
2,350	5	2	45	500
1,600	3	2	20	450
2,000	3	2	30	250
854	2	1	10	200
560	1	1	30	180

Multiple Linear Regression: Training

- Given:

$$\{(x^{(1)}, y^{(1)}), (x^{(2)}, y^{(2)}), \ldots, (x^{(m)}, y^{(m)})\}$$

- We want:

$$\hat{y}^{(i)} \approx y^{(i)}$$

Multiple Linear Regression: Training

- Loss Function: the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}(y^{(i)} - \hat{y}^{(i)})^2$$

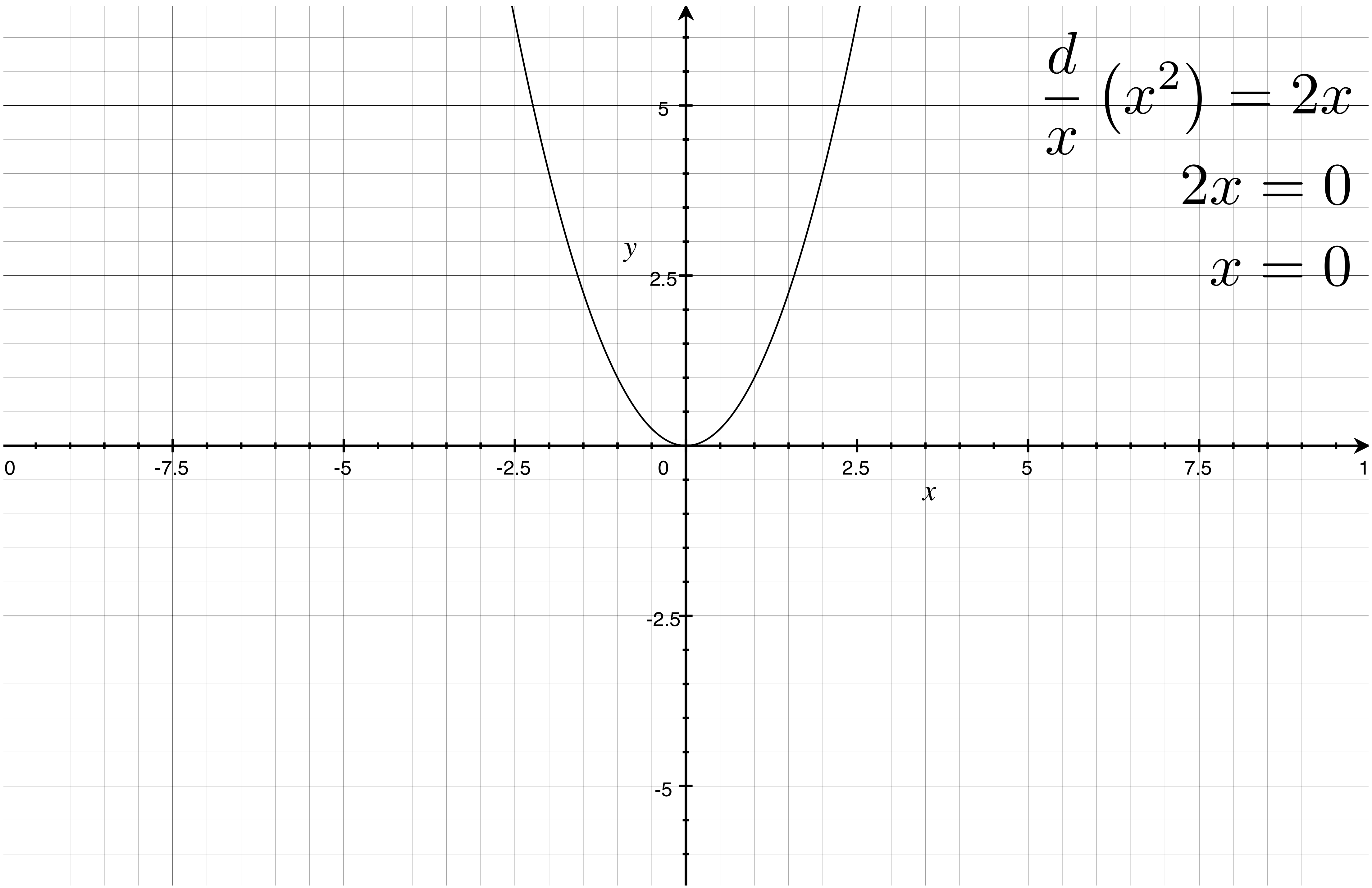
Multiple Linear Regression: Training

- Cost Function: the discrepancy between the predicted and actual output values for all training instances

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}(y^{(i)} - \hat{y}^{(i)})^2$$

$$\mathcal{J}(w, b) = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{1}{2}(y^{(i)} - \hat{y}^{(i)})^2\right)$$

Derivatives

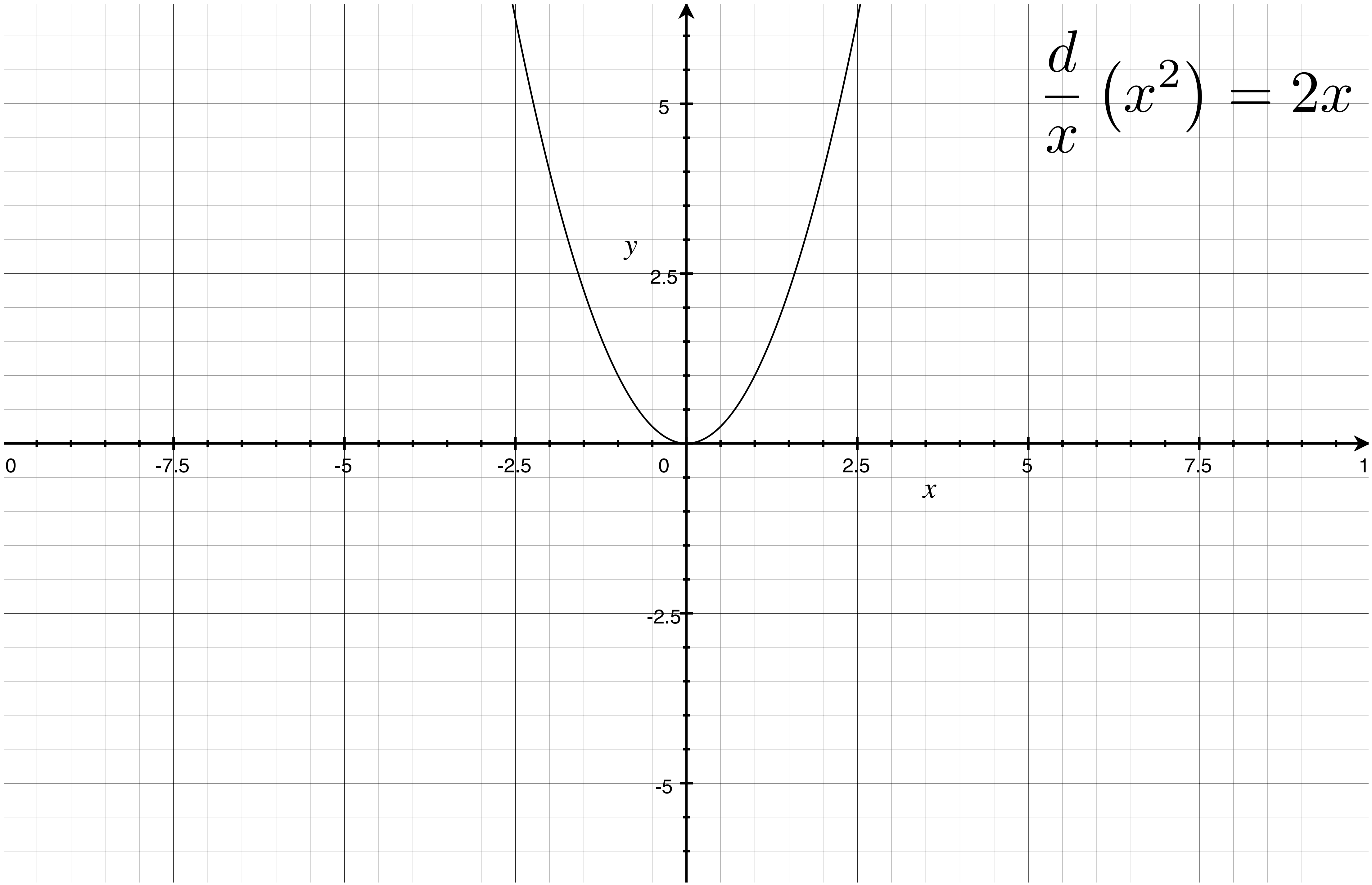


$$\frac{d}{x}\left(x^2\right) = 2x$$

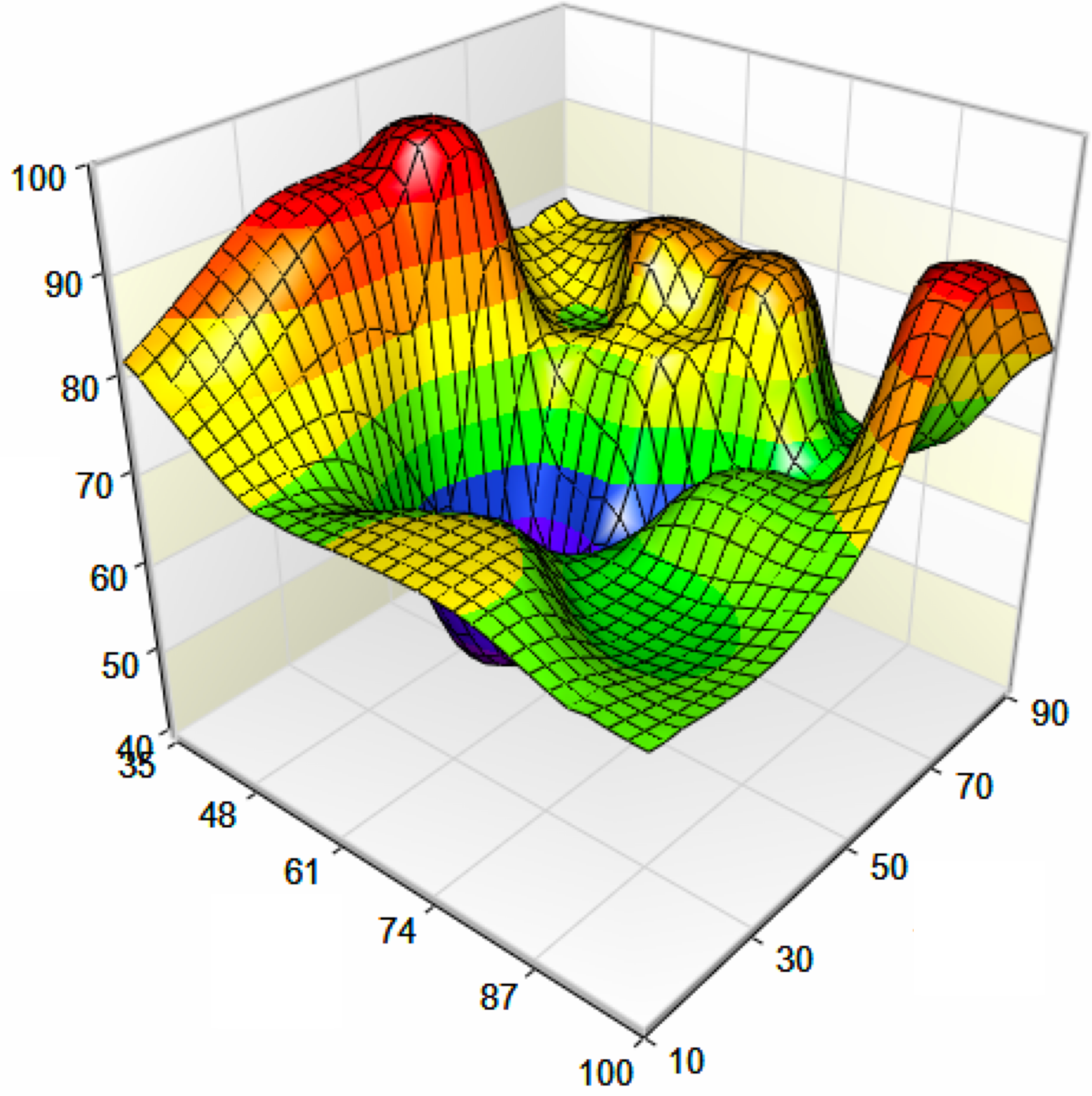
$$2x = 0$$

$$x = 0$$

Gradient Descent: Intuition



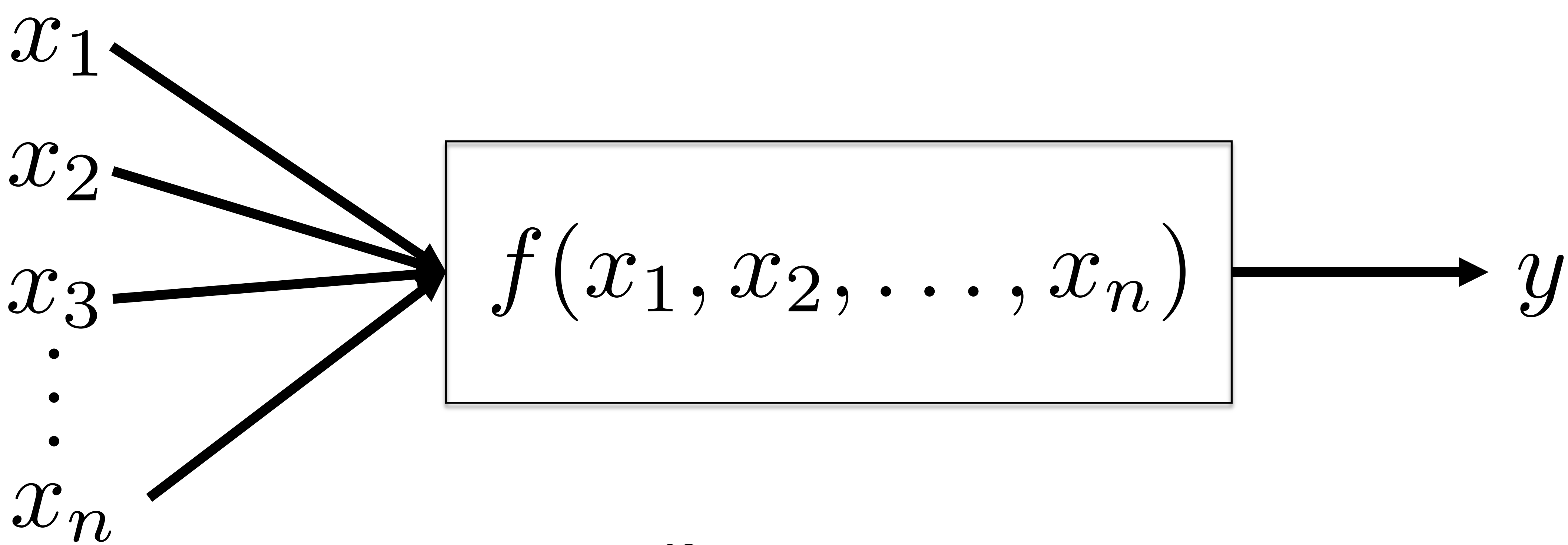
Gradient Descent: Intuition



Multiple Linear Regression

Size (feet)	No. of bedrooms	No. of floors	Age (years)	Price (x$1000)
2,350	5	2	45	500
1,600	3	2	20	450
2,000	3	2	30	250
854	2	1	10	200
560	1	1	30	180

Multiple Linear Regression



$$y = \sum_{j=1}^{n} (w_j x_j) + b$$

Gradient Descent

- Loss Function: the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}(y^{(i)} - \hat{y}^{(i)})^2$$

- Let's see what the slope of the loss function is with respect to parameter *b*!
- **Note:** this will only consider one training example!

Gradient Descent

- Derivative of the loss function with respect to b

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}(y^{(i)} - \hat{y}^{(i)})^2$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}\left(y^{(i)} - \sum_{j=1}^{n}(w_j x_j^{(i)}) - b\right)^2$$

$$\frac{d}{db}\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = -\left(y^{(i)} - \hat{y}^{(i)}\right)$$

$$\frac{d}{db}\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \hat{y}^{(i)} - y^{(i)}$$

Gradient Descent

$$\frac{d}{db}\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \hat{y}^{(i)} - y^{(i)}$$

Scenario	$\hat{y}^{(i)} - y^{(i)}$	Action!
$\hat{y}^{(i)} > y^{(i)}$	+	Decrease (nudge left)
$\hat{y}^{(i)} < y^{(i)}$	--	Increase (nudge right)
$\hat{y}^{(i)} \approx y^{(i)}$	0	Do nothing!

$$y = \sum_{j=1}^{n}(w_j x_j) + b$$

Gradient Descent

- Loss Function: the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}(y^{(i)} - \hat{y}^{(i)})^2$$

- Let's see what the slope of the loss function is with respect to parameter w_j!
- **Note:** this will only consider one training example!

Gradient Descent

- Derivative of the loss function with respect to w_j

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}(y^{(i)} - \hat{y}^{(i)})^2$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}\left(y^{(i)} - \sum_{j=1}^{n}(w_j x_j^{(i)}) - b\right)^2$$

$$\frac{d}{dw_j}\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = -x_j^{(i)}\left(y^{(i)} - \hat{y}^{(i)}\right)$$

$$\frac{d}{dw_j}\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = (\hat{y}^{(i)} - y^{(i)})x_j$$

Gradient Descent

$$\frac{d}{dw_j}\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = (\hat{y}^{(i)} - y^{(i)})x_j$$

Scenario	$\hat{y}^{(i)} - y^{(i)}$	Action!
$\hat{y}^{(i)} > y^{(i)}$	+	Go in the opposite direction as $x_j^{(i)}$
$\hat{y}^{(i)} < y^{(i)}$	-	Go in the same direction as $x_j^{(i)}$
$\hat{y}^{(i)} \approx y^{(i)}$	0	Do nothing!

$$y = \sum_{j=1}^{n}(w_j x_j) + b$$

Gradient Descent

- Loss Function: the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \frac{1}{2}(y^{(i)} - \hat{y}^{(i)})^2$$

- Given one training example, we can take derivatives with respect to each parameter to see what direction we should be going to minimize the loss function.

Gradient Descent

- Repeat many times (or until convergence):

$$b \leftarrow b - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(\hat{y}^{(i)} - y^{(i)} \right)$$

$$w_j \leftarrow w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} \left((\hat{y}^{(i)} - y^{(i)}) x_j^{(i)} \right)$$

Gradient Descent

- Repeat many times (or until convergence):

$$b \leftarrow b - \alpha \boxed{\frac{1}{m} \sum_{i=1}^{m} \left(\hat{y}^{(i)} - y^{(i)} \right)}$$

- If we are overshooting the target, reduce b
- If we are undershooting the target, increase b
- Otherwise, do nothing

Gradient Descent

- Repeat many times (or until convergence):

$$w_j \leftarrow w_j - \alpha \boxed{\frac{1}{m} \sum_{i=1}^{m} \left((\hat{y}^{(i)} - y^{(i)}) x_j^{(i)} \right)}$$

- If we are overshooting the target, reduce w_j proportional to the value of x_j
- If we are undershooting the target, increase w_j proportional to the value of x_j
- Otherwise, do nothing

Overview

- Philosophical questions
- Derivatives: What are they good for?
- Linear regression
- Multiple linear regression
- Logistic regression

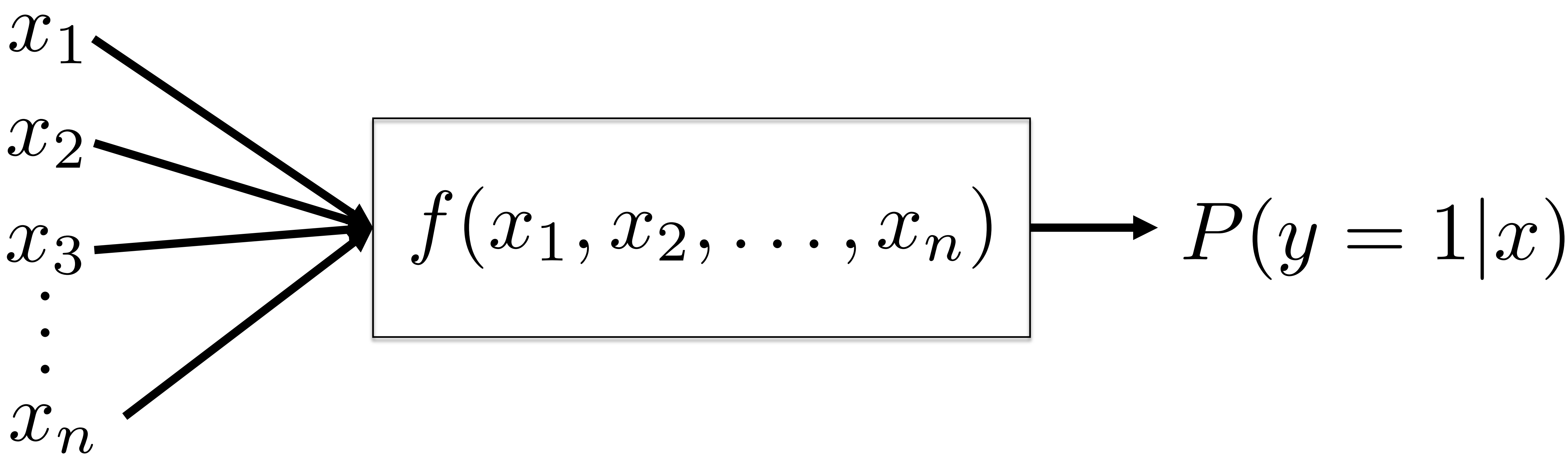
Logistic Regression

- Linear regression: predict *y* given *x*
- Multiple linear regression: predict *y* given *x_1, x_2, …, x_n*

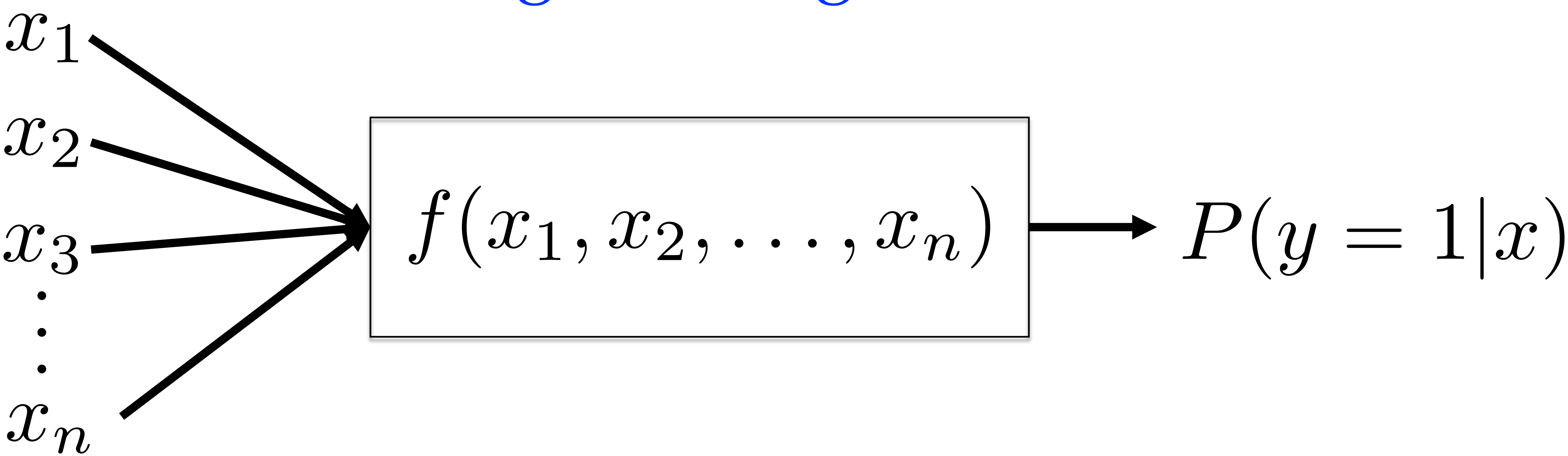
Logistic Regression

- Logistic Regression: predict $P(y=1|x_1, x_2, \ldots, x_n)$
- We can use logistic regression to do binary classification.

Logistic Regression



Logistic Regression

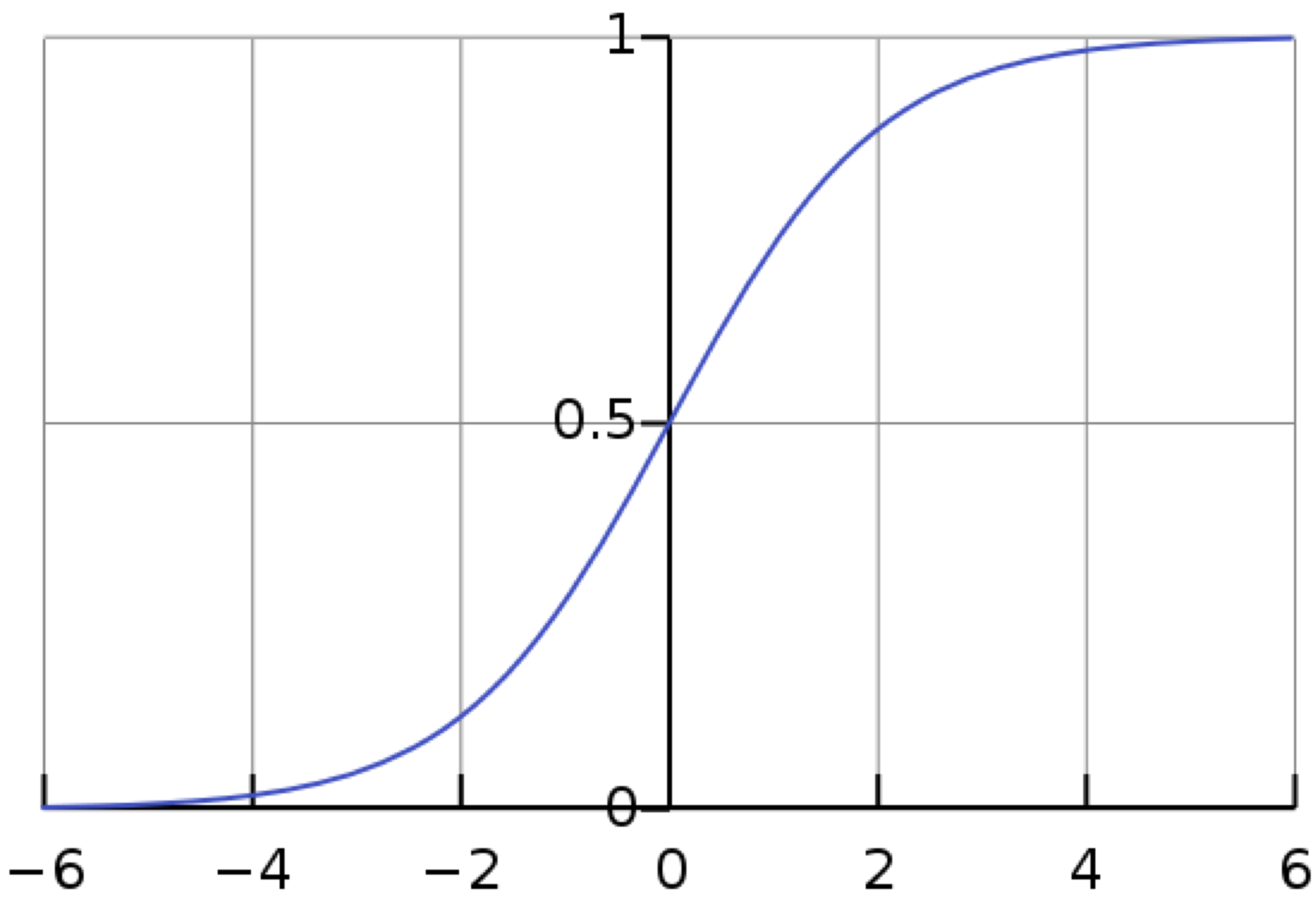


$$\sigma(z) = \frac{1}{1 + e^{-z}}$$

$$z = \sum_{j=1}^{n} (w_j x_j) + b$$

Logistic Regression

$$\sigma(z) = \frac{1}{1 + e^{-z}}$$



Logistic Regression

- Loss Function: the discrepancy between the predicted and actual output values for a single training instance

$$z = \sum_{j=1}^{n} (w_j x_j) + b$$

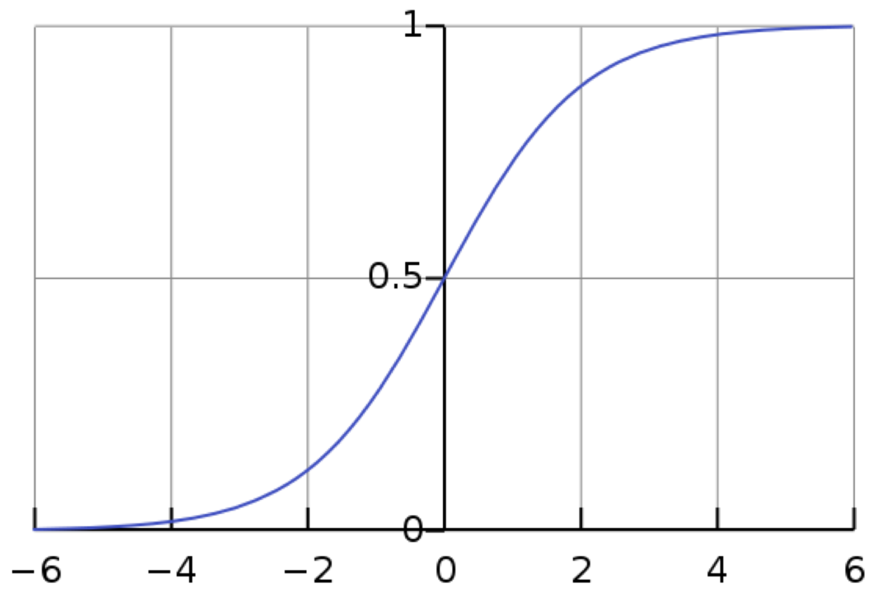
$$\hat{y} = \sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = -\Big(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)})\Big)$$

Logistic Regression

- Loss Function: the discrepancy between the predicted and actual output values for <u>a single</u> training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = -\left(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)})\right)$$

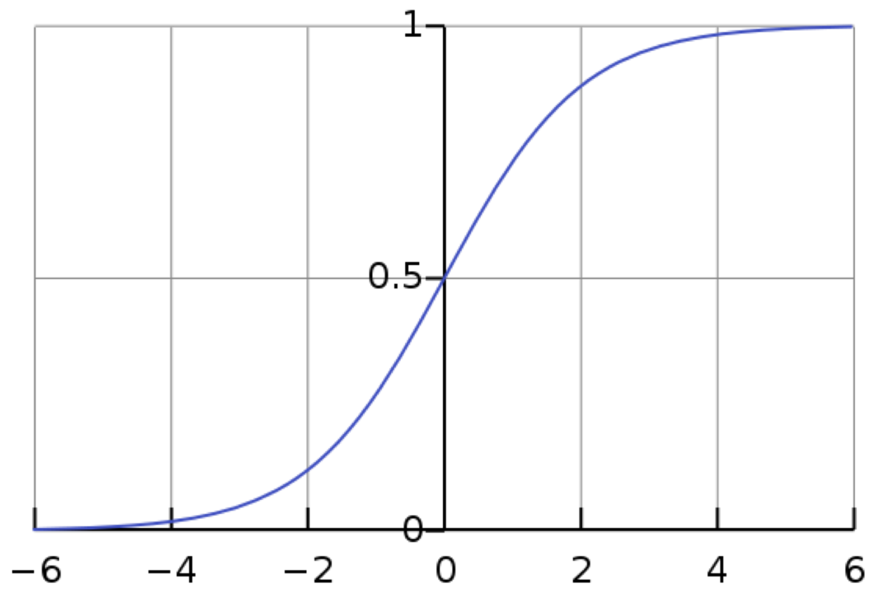


- If the true value is 1, we want the predicted value to be <u>high</u>.
- Remember: log(1) = 0

Logistic Regression

- Loss Function: the discrepancy between the predicted and actual output values for a single training instance

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = -\left(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)})\right)$$



- If the true value is 0, we want the predicted value to be low.
- Remember: log(1) = 0

Logistic Regression

- Loss Function: the discrepancy between the predicted and actual output values for a single training instance

$$z = \sum_{j=1}^{n} (w_j x_j) + b$$

$$\hat{y} = \sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = -\left(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)}) \right)$$

$$\frac{d}{db}\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = \hat{y}^{(i)} - y^{(i)}$$

Logistic Regression

- Loss Function: the discrepancy between the predicted and actual output values for a single training instance

$$z = \sum_{j=1}^{n} (w_j x_j) + b$$

$$\hat{y} = \sigma(z) = \frac{1}{1 + e^{-z}}$$

$$\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = -\Big(y^{(i)} \log \hat{y}^{(i)} + (1 - y^{(i)}) \log(1 - \hat{y}^{(i)})\Big)$$

$$\boxed{\frac{d}{dw_j}\mathcal{L}(y^{(i)}, \hat{y}^{(i)}) = (\hat{y}^{(i)} - y^{(i)})x_j}$$

Logistic Regression:
Gradient Descent

- Repeat many times (or until convergence):

$$b \leftarrow b - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(\hat{y}^{(i)} - y^{(i)} \right)$$

$$w_j \leftarrow w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} \left((\hat{y}^{(i)} - y^{(i)}) x_j^{(i)} \right)$$

Logistic Regression:
Gradient Descent

- Repeat many times (or until convergence):

$$b \leftarrow b - \alpha \frac{1}{m} \sum_{i=1}^{m} \left(\hat{y}^{(i)} - y^{(i)} \right)$$

- If we are overshooting the target, reduce b
- If we are undershooting the target, increase b
- Otherwise, do nothing

Logistic Regression:
Gradient Descent

- Repeat many times (or until convergence):

$$w_j \leftarrow w_j - \alpha \frac{1}{m} \sum_{i=1}^{m} \Big((\hat{y}^{(i)} - y^{(i)}) x_j^{(i)} \Big)$$

- If we are overshooting the target, reduce w_j proportional to the value of x_j
- If we are undershooting the target, increase w_j proportional to the value of x_j
- Otherwise, do nothing

Overview

- Philosophical questions
- Derivatives: What are they good for?
- Linear regression
- Multiple linear regression
- Logistic regression

The Big Picture!

- Linear regression, multiple linear regression and logistic regression are examples of linear models
- Internally, linear models output a prediction based on a weighted combination of input features
- Features that are <u>positively correlated</u> with the target output get a <u>high</u> weight
- Features that are <u>negatively correlated</u> with the target output get a <u>negative</u> weight
- Features that are <u>uncorrelated</u> with the target output get a <u>zero</u> weight